Time Integration of the Equations of Motion of a Structural System Including Damping

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The conditions required to achieve the unconditional spectral stability for damped linear elastic structural systems are determined in closed form for several linear single-step implicit temporal operators. The results obtained include the effect of system damping on algorithmic damping, on relative period errors, and on the spectral radius of each operator. It is found that inclusion of system damping results in more algorithmic damping in the low-frequency modes than in the high-frequency modes and it also causes a tendency to overshoot the exact response as compared to the undamped case. Based on the above, recommendations are made with regard to the choice of integration parameters for each operator.

Introduction

CONSIDERABLE research has been presented dealing with the integration of linear semidiscretized equations of motion,[1-12] but no rigorous analytic treatment of the damped problem has been reported in the open literature. For many aerospace applications it is imperative that damping be included in the modeling process and it has been illustrated that the linear viscous damping model is completely adequate for lightly damped structures[13] but that unfortunately the damping matrix will not always be proportional. The present work addresses the problem of applying linear single-step temporal operators to discretized motion equations with nonproportional damping matrices. A rigorous closed-form analysis of the stability and accuracy characteristics of the operators when applied to the generally damped motion equations is presented for the first time, although some of the present conclusions have been arrived at elsewhere.[14]

Because the results of this investigation are destined for use with large finite element models, unconditional stability of the temporal operator is considered to be essential. This restriction eliminates all explicit methods[1] and, due to the considerable computational effort at each time step, only single-step implicit methods are considered. The use of a single-step method also eliminates the need for a special starting algorithm that would require extra storage and could be a potential source of inaccuracies or instabilities.[5] Implicit methods may also be second-order accurate, which is of some importance since these methods are superior to first-order methods for linear structural dynamics problems[2-5] and involve no computational penalty. Further, methods that display a potential for controllable dissipation in the high-frequency modes will be preferred in multi-degree-of-freedom systems since the question of accuracy gives rise to a unique dilemma: if the step size is small enough to integrate all of the modes accurately, then it is too small to be computationally efficient for the low-frequency modes; however, if it is large enough to be efficient for the low-frequency modes, then the method will be inaccurate for the high-frequency modes. This problem is overcome by employing integration procedures that possess numerical dissipation, the presence of which allows the temporal operator to provide a solution that has the high-frequency contributions filtered out, leaving an accurate representation of the primary or low-frequency response.[2-4]

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Stability with Nonproportional Damping

The motion equations corresponding to a discretized structural system can be expressed as

$$M\ddot{X}+C\dot{X}+KX=R(t) \tag{1}$$

In the linear case, M, C, and K are constant and represent the mass, damping, and stiffness matrices, respectively, while X is the vector of displacements and $R(t)$ the vector of time-varying external loads. Time derivatives are denoted by superposed dots.

It will be now be shown that multi-degree-of-freedom systems with nonproportional damping matrices can be successfully integrated by any temporal operator that is consistent and unconditionally stable for the damped single-degree-of-freedom problem.

Proceeding in the manner described by Foss[15] let

$$Z=[\dot{X}(t),X(t)]^T \tag{2}$$

which allows the discretized motion equations to be expressed as

$$G\dot{Z}+HZ=P(t) \tag{3}$$

where

$$G=\begin{bmatrix} 0 & M \\ M & C \end{bmatrix};\quad H=\begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix};\quad P(t)=\begin{bmatrix} 0 \\ R(t) \end{bmatrix} \tag{4}$$

and where it is noted that G and H are symmetric since M, C, and K are symmetric.

The eigenvalue problem corresponding to Eq. (3) is

$$\lambda G\Phi+H\Phi=0 \tag{5}$$

and the generally complex eigenvalues and eigenvectors are denoted by λ_j and Φ_j, respectively, and where it is noted that the eigenvector Φ_j takes the form

$$\Phi_j=[\lambda_j\bar{\Phi}_j,\bar{\Phi}_j]^T \tag{6}$$

due to the definition of Z. For the solution of Eq. (3), Z is expressed in the form

$$Z(t)=\sum_{j=1}^{2N}Q_j(t)\Phi_j \tag{7}$$

where $Q(t)$ are time-varying scalar coefficients and N the dimension of M, C, and K.

Following usual procedures and making use of the orthogonal character of Φ_j yields

$$\dot{Q}_j - \lambda_j Q_j = P_j(t); \qquad j = 1, 2N \tag{8}$$

where

$$\lambda_j = -\frac{\Phi_j^T H \Phi_j}{\Phi_j^T G \Phi_j} \qquad P_j(t) = \frac{\Phi_j^T P(t)}{\Phi_j^T G \Phi_j} \tag{9}$$

and where it is noted that Q_j, λ_j, and P_j are, in general, complex. Further, for every equation where these quantities are complex, there exists the complex conjugate to the above

$$\dot{Q}_j^* - \lambda_j^* Q_j^* = P_j^*(t) \tag{10}$$

where $(\ )^*$ denotes a complex conjugate.

Equation (8) can be altered to a differential equation involving real variables by making the substitutions

$$Q_j = x_j + iy_j, \quad \lambda_j = -\xi_j\omega_j + i\omega_j\sqrt{1-\xi_j^2}, \quad P_j = p_j + if_j \tag{11}$$

to obtain

$$\dot{x}_j + i\dot{y}_j + (\xi_j\omega_j - i\omega_j\sqrt{1-\xi_j^2})(x_j + iy_j) = p_j + if_j \tag{12}$$

and a similar expression for the complex conjugate.

In the above ξ_j and ω_j are the damping ratio and undamped natural frequency for the jth mode respectively. Separating the real and imaginary parts in either of these equations gives

$$\dot{x}_j + \xi_j\omega_j x_j + \omega_j\sqrt{1-\xi_j^2}\, y_j = p_j \tag{13}$$

$$\dot{y}_j - \omega_j\sqrt{1-\xi_j^2}\, x_j + \xi_j\omega_j y_j = f_j \tag{14}$$

These equations can be uncoupled. Doing so results in

$$\ddot{x}_j + 2\xi_j\omega_j\dot{x}_j + \omega_j^2 x_j = \dot{p}_j - \omega_j\sqrt{1-\xi_j^2}\, f_j + \omega_j\xi_j p_j \tag{15}$$

$$\ddot{y}_j + 2\xi_j\omega_j\dot{y}_j + \omega_j^2 y_j = \dot{f}_j + \omega_j\sqrt{1-\xi_j^2}\, p_j + \omega_j\xi_j f_j \tag{16}$$

The above equations for x_j and y_j are second-order, real differential equations and the response variable $Z(t)$ is composed of a finite sum of terms involving linear combinations of x_j and y_j and the eigenvectors Φ_j. Thus, since the $\|\Phi_j\|$ are bounded, then $Z(t)$ will be bounded if x_j and y_j are bounded. Therefore, in the investigation of the stability of the numerical algorithm it is sufficient to investigate the stability of the algorithm with regard to the differential operator

$$(\ddot{\ }) + 2\xi_j\omega_j(\dot{\ }) + \omega_j^2(\ ) \tag{17}$$

On this basis, it follows that the consistency and stability characteristics of an integration scheme may be considered on the basis of the operator [Eq. (17)]. This is an important result since it removes the restriction that only proportional damping matrices are considered. All further work will use the following form of the single-degree-of-freedom motion equation,

$$a + 2\xi\omega v + \omega^2 d = F(t) \tag{18}$$

where a, v, and d are the acceleration, velocity, and displacement, respectively, while $F(t)$ is the time-varying applied load.

In the sequel, four methods of solving this motion equation will be considered: the Newmark-β,[7] the Wilson-θ collocation,[8] the Hilber-Hughes collocation,[2] and the Hilber-Hughes-α.[3] In the study, the collocation methods will be considered first, followed by the alpha methods.

It is to be noted that present operator analysis does not address the question of convergence directly, even though it is probably one of the most important results of the work. This procedure is possible because, for the operators under discussion here, the Lax equivalence theorem[9] guarantees convergence provided that stability and consistency are assured.

Collocation Schemes

In order to remove the overshoot characteristics of the Wilson-θ operator,[4,8] a more general formulation was developed in Ref. 2. The generalization will be presented here, with the Wilson-θ method being treated as a special case.

General Formulation

The motion equation is true for all time t. Therefore it must be satisfied at $t = (n+\theta)\Delta t$ (as designated by the subscripts) where Δt is the time-step size, n a positive integer, and θ the collocation parameter. The corresponding motion equation is

$$a_{n+\theta} + 2\xi\omega v_{n+\theta} + \omega^2 d_{n+\theta} = F_{n+\theta} \tag{19}$$

The difference equations used to solve Eq. (19) are[2]

$$a_{n+\theta} = (1-\theta)a_n + \theta a_{n+1} \tag{20}$$

$$v_{n+\theta} = v_n + \theta\Delta t[(1-\gamma)a_n + \gamma a_{n+\theta}] \tag{21}$$

$$d_{n+\theta} = d_n + \theta\Delta t v_n + \theta^2\Delta t^2[(½-\beta)a_n + \beta a_{n+\theta}] \tag{22}$$

where γ and β are parameters chosen to control the accuracy and stability of the algorithm.

By substituting these expressions for $a_{n+\theta}$, $v_{n+\theta}$, and $d_{n+\theta}$ into Eq. (19), an expression for a_{n+1} in terms of a_n, v_n, and d_n can be obtained.

Then, setting $\theta = 1$ in Eqs. (21) and (22) and using the previously obtained expression for a_{n+1}, the values of v_{n+1} and d_{n+1} in terms of the displacement, velocity, and acceleration at time $n\Delta t$ may be found.

Doing so yields a linear one-step method, expressible in the form

$$X_{n+1} = AX_n + LF_{n+1} \tag{23}$$

where the displacement vector X has been defined as $X_n = [d_n, \Delta t v_n, \Delta t^2 a_n]^T$.[2-4] In the above, A is the integration approximation operator and L the load approximation operator. For the present operator the approximation matrix A takes the form

$$A = \begin{bmatrix} 1-\beta\dfrac{\Omega^2}{D} & 1-2\beta\xi\dfrac{\Omega}{D}-\beta\theta\dfrac{\Omega^2}{D} & \dfrac{1}{2}-\dfrac{\beta}{D}\left(1+2\xi\theta\Omega+\dfrac{1}{2}\theta^2\Omega^2\right) \\ -\gamma\dfrac{\Omega^2}{D} & 1-2\gamma\xi\dfrac{\Omega}{D}-\gamma\theta\dfrac{\Omega^2}{D} & 1-\dfrac{\gamma}{D}\left(1+2\xi\theta\Omega+\dfrac{1}{2}\theta^2\Omega^2\right) \\ -\dfrac{\Omega^2}{D} & -2\xi\dfrac{\Omega}{D}-\theta\dfrac{\Omega^2}{D} & 1-\dfrac{1}{D}\left(1+2\xi\theta\Omega+\dfrac{1}{2}\theta^2\Omega^2\right) \end{bmatrix} \tag{24}$$

where

$$D = [\theta + 2\gamma\xi\theta^2\Omega + \beta\theta^3\Omega^2] \quad \text{and} \quad \Omega = \omega\Delta t = 2\pi\Delta t/T \tag{25}$$

These expressions reduce to those given in Ref. 2 when $\xi = 0$.

The integration technique is unconditionally stable for the homogeneous form of Eq. (19) if the solution for any initial value X_0 is bounded for any step size Δt (even when Δt is large in relation to the period T). This stability question is answered by examining the magnitude of the eigenvalues of A as a function of Ω. Specifically A is spectrally stable if its spectral radius $\rho(A)$ satisfies $\rho(A) \leq 1$ (the equality does not apply for repeated eigenvalues) where $\rho(A) \triangleq \max |\lambda_j|$ and λ_j (generally complex), $j=1,3$, are the eigenvalues of A.[16] The characteristic equation is

$$\lambda^3 - 2A_1\lambda^2 + A_2\lambda - A_3 = 0 \tag{26}$$

where A_j; $j=1,3$, the operator invariants, are defined by

$$2A_1 = \text{trace } A, \quad A_2 = \sum_{j=1}^{3} M_{jj} \quad \text{and} \quad A_3 = \det A \tag{27}$$

where M_{jj} are the principal minors of A.

In order to determine the constraints on A_1, A_2, and A_3 such that the λ_j lie within the unit circle, the transformation

$$\lambda = (1+z)/(1-z) \tag{28}$$

is made, which maps the unit circle in the λ plane into the left half of the z plane while the boundary $|\lambda| = 1$ becomes the imaginary axis.[17] Thus, the characteristic equation takes the form

$$\frac{1}{(1-z)^3}[(1+2A_1+A_2+A_3)z^3 + (3+2A_1-A_2-3A_3)z^2 + (3-2A_1-A_2+3A_3)z + (1-2A_1+A_2-A_3)] = 0 \tag{29}$$

and spectral stability is achieved when all roots lie within the left half plane. This implies that $z=1$ is not of interest and only the numerator of Eq. (29) need be considered. Furthermore, the constraints on the equation coefficients can be investigated using the Hurwitz criterion, which yields the necessary and sufficient conditions that the roots have only negative real parts.[17]

The stability conditions so obtained are

$$(1-2A_1+A_2-A_3) > 0 \tag{30}$$

$$(1+2A_1+A_2+A_3) > 0 \tag{31}$$

$$(3+2A_1-A_2-3A_3) > 0 \tag{32}$$

$$(3-2A_1-A_2+3A_3) > 0 \tag{33}$$

$$(1-A_2+2A_1A_3-A_3^2) > 0 \tag{34}$$

where it is noted that satisfaction of these conditions for all Ω will guarantee unconditional stability.

A second concern is the accuracy of the operator. Here the motion equation and its corresponding difference equation are said to be consistent to an order of accuracy k, if the local truncation error σ is of the form $\sigma = 0(\Delta t^k)$ for $k>0$. The local truncation error is obtained by applying Eq. (23) recursively (with the applied load set to zero) to form a difference equation in terms of displacements only. This manipulation results in the equation

$$d_{n+1} - 2A_1 d_n + A_2 d_{n-1} - A_3 d_{n-2} = 0 \tag{35}$$

with the corresponding truncation error[2]

$$\sigma = (1/\Delta t^2)[u(t+\Delta t) - 2A_1 u(t) + A_2 u(t-\Delta t) - A_3 u(t-2\Delta t)] \tag{36}$$

where u satisfies the homogeneous motion equation. In order to obtain the truncation error dependence on Δt, each term involving u in Eq. (36) is expanded in Taylor series about t. Then the local truncation error takes the form

$$\sigma = \sum_{i=0}^{m} T_i \Delta t^{i-2} u^{(i)}(t) + 0(\Delta t^{m-1}) \tag{37}$$

where the superscript i denotes the ith derivative with respect to t and where

$$T_0 = 1 - 2A_1 + A_2 - A_3 \tag{38}$$

and

$$T_i = [1 + (-1)^i A_2 - (-2)^i A_3]\frac{1}{i!}, \qquad i>0 \tag{39}$$

The fact that $u(t)$ satisfies the motion equation is then used to eliminate second- and higher-order derivatives of $u(t)$. Substitution for the values of the operator invariants,

$$A_1 = 1 - \frac{1}{2D}\left[2\xi\Omega + \Omega^2\left(\theta+\gamma-\frac{1}{2}\right)\right] + \frac{1}{2}A_3 \tag{40}$$

$$A_2 = 1 - \frac{1}{D}\left[2\xi\Omega - \Omega^2\left(\theta+\gamma-\frac{3}{2}\right)\right] + 2A_3 \tag{41}$$

$$A_3 = \frac{1}{D}\left[(\theta-1) + 2\xi\Omega(1-\theta-\gamma+\gamma\theta^2) + \Omega^2(\theta-1) \times \left(\beta(\theta^2+\theta+1) - \gamma - \frac{1}{2}(\theta-1)\right)\right] \tag{42}$$

yields an expression for the truncation error as

$$\sigma = \frac{\Omega\omega}{D}\left\{[\xi(1-2\gamma)\omega u] + \left[\left(\left(\gamma-\frac{1}{2}\right) + 2\xi^2(1-2\gamma)\right)\dot{u}\right]\right\} + 0(\Delta t^2) \tag{43}$$

In Eq. (43) it is noted that $m=3$ [Eq. (37)] has been employed. Furthermore, second-order accuracy is achieved when $\gamma = 1/2$, which is identical to the result for the undamped case.

Since attention is being restricted to operators exhibiting second-order accuracy and unconditional stability, the remainder of the analysis examines the conditions on θ and β that yield unconditional stability when $\gamma = 1/2$. Application of the stability conditions shows that the general collocation scheme is unconditionally stable for

$$\theta \geq 1 \quad \text{and} \quad (2\theta^2-1)/[4(2\theta^3-1)] \leq \beta \leq \theta/(2\theta+2) \tag{44}$$

where it has been implicitly assumed that $\xi \geq 0$. These results are in complete agreement with those in Ref. 2 for the undamped case. The present stability domain is illustrated in Fig. 1.

It is to be noted that the present results reduce to the Newmark family of operators when $\theta = 1$ and that the further restrictions $\gamma = 1/2$ and $\beta = 1/4$ yield the average acceleration method. Also, the Wilson-θ operator can be deduced by setting $\gamma = 1/2$ and $\beta = 1/6$. This further implies that the Wilson-θ technique is second-order accurate for the damped case as well as the undamped case. Another point of importance is that when $\gamma = 1/2$ and $\beta = 1/6$, the stability conditions [Eqs. (30-34)] imply that the Wilson-θ operator will be unconditionally stable for damped and undamped systems when $\theta \geq (1+\sqrt{3})/2$.

For sufficiently large Ω, the terms in the A matrix involving damping become negligible and correspondingly the behavior of the spectral radius as $\Omega \to \infty$ is independent of the damping ratio ξ. This is shown in Fig. 2 and implies that the optimal parameter pairs as determined in Ref. 2 are valid for the damped as well as the undamped problem. These values are plotted in Fig. 1 for convenience.

The results shown in Fig. 2 are representative of the behavior of the general collocation operator's spectral radius and nonzero values of system damping for the optimal

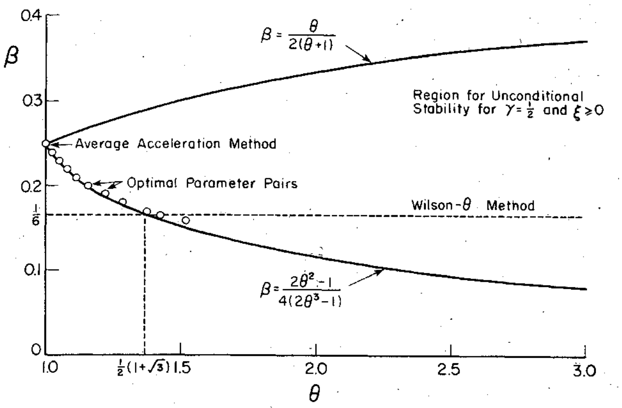

Fig. 1 General collocation method stability boundaries.

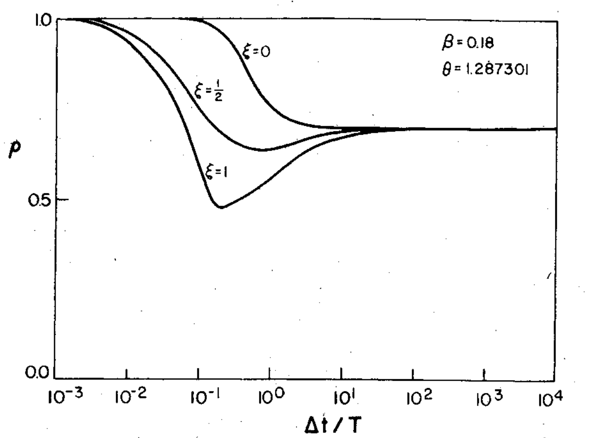

Fig. 2a General collocation method spectral radius behavior for a representative optimal parameter pair.

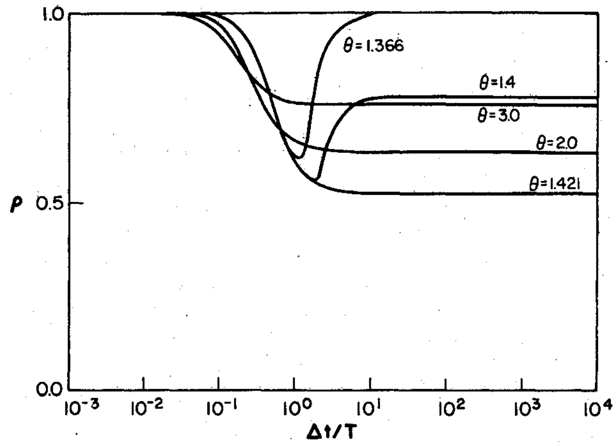

Fig. 2b Wilson-θ method spectral radius behavior, ξ = 0.

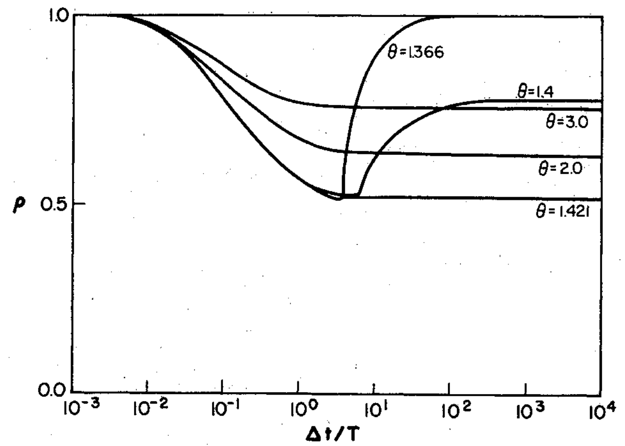

Fig. 2c Wilson-θ method spectral radius behavior, ξ = 0.5.

parameter pairs. The presence of system damping yields an undesirable dipping behavior shown at intermediate values of $\Delta t/T$. Furthermore, while superior spectral properties are achieved with decreasing β, there is a penalty exacted in the form of increased algorithmic damping and relative period error. A compromise between these two conflicting properties must be made in choosing a value of β.

The collocation methods have a predictable algorithmic damping ratio, $\bar{\xi}$, as $\Omega \to 0$, namely

$$\lim_{\Omega \to 0} \bar{\xi} = \xi \tag{45}$$

This is evident in Fig. 3a where it is observed that for the damped problem the low-frequency modes are damped more than the high-frequency modes and for the undamped problem the greatest algorithmic damping is observed in the high-frequency modes. In addition, for the second-order operators it appears from numerical experiment that the maximum value of the algorithmic damping ratio occurs at $\Omega = 0$ when $\xi > 0$.

Both the algorithmic damping ratio and the relative period error decrease with increasing β and this trend is maintained regardless of the magnitude of ξ. Also, it was found that changes in integration parameters had a more pronounced effect on the error measures than changes in system damping magnitude.

An indication of the overshoot characteristics of the collocation operator can be obtained by using the method outlined in Ref. 2, where it is shown that the overshoot characteristics can be determined by calculating the response at the end of the first time step as $\Omega \to \infty$. This yields

$$d_1 \sim O(\Omega^2) d_0 + O(\Omega^2)(\xi/\omega) v_0 + O(\Omega) v_0/\omega \tag{46}$$

$$v_1 \sim O(\Omega)\omega d_0 + O(\Omega)\xi v_0 + O(1) v_0 \tag{47}$$

It is noted that these results are identical to those of the Wilson-θ operator. Furthermore, it follows that system damping does not improve overshoot behavior and that collocation methods possess inherent quadratic overshoot of the displacement for all values of ξ.

This tendency to significantly overshoot the exact solution of the initial value problem will manifest itself most strongly in the high-frequency modes of a multi-degree-of-freedom problem. Therefore, it may be concluded that collocation methods are ill-suited for problems involving impact or rapidly varying loads where the high-frequency components of the response are extremely important.

Wilson-θ Results

When the Wilson-θ method is used for undamped systems it has been shown[10] that there is only one value of θ which results in the principal roots of the characteristic equation having double zeros. Further, this value marks a bifurcation from a single to a pair of real roots as θ is further decreased. The value of θ for which this result is achieved is

$$\bar{\theta} = \frac{1}{2} + \left[\frac{7 + (32)^{1/3}}{12}\right]^{1/2} \cong 1.4208149517 \tag{48}$$

It has been demonstrated numerically that this value of θ yields the minimum value of the spectral radius as $\Omega \to \infty$.[3,13] Using the rule of thumb that dictates 10 time steps per period, the Wilson-θ method with θ chosen equal to $\bar{\theta}$ yields a percentage relative period error that is not appreciably different from the error given for $\theta = 1.4$.[3] The value of $\theta = \bar{\theta}$ thus gives superior dissipation and spectral properties[2] with no penalty in accuracy relative to $\theta = 1.4$.

The presence of system damping does not change the value of θ which gives the minimum spectral radius, but it does change the dependence of the spectral radius on $\Delta t/T$ as may be seen in Fig. 2. Numerical studies indicate that the spectral radius is a local minimum for values of $\Delta t/T$ near unity and that this minimum value is not altered by the introduction of system damping. For $\xi > 0$ the spectral radius attains a value very close to the minimum for successively smaller values of $\Delta t/T$ as ξ increases (see Fig. 2).

Alpha Methods

The α algorithm is based on the following set of difference relations,[3]

$$a_{n+1} + 2\xi\omega v_{n+1} + (1+\alpha)\omega^2 d_{n+1} - \alpha\omega^2 d_n = F_{n+1} \quad (49)$$

$$v_{n+1} = v_n + \Delta t[(1-\gamma)a_n + \gamma a_{n+1}] \quad (50)$$

$$d_{n+1} = d_n + \Delta t v_n + \Delta t^2[(\tfrac{1}{2}-\beta)a_n + \beta a_{n+1}] \quad (51)$$

where α, β, and γ are the integration parameters governing the stability and accuracy of the operator.

Proceeding as before and substituting the expressions for v_n and d_n into Eq. (49), an equation for a_{n+1} in terms of a_n, v_n, and d_n is obtained. Back substituting into Eqs. (50) and (51) allows v_{n+1} and d_{n+1} to be expressed solely in terms of a_n, v_n, and d_n. This yields the matrix representation as before with

$$A = \frac{1}{D}\begin{bmatrix} 1+2\xi\Omega\gamma+\alpha\beta\Omega^2 & 1+2\xi\Omega(\gamma-\beta) & (\tfrac{1}{2}-\beta)+2\xi\Omega(\tfrac{1}{2}\gamma-\beta) \\ -\gamma\Omega^2 & 1+(1+\alpha)(\beta-\gamma)\Omega^2 & (1-\gamma)+(1+\alpha)(\beta-\tfrac{1}{2}\gamma)\Omega^2 \\ -\Omega^2 & -2\xi\Omega-(1+\alpha)\Omega^2 & -2\xi\Omega(1-\gamma)-(1+\alpha)(\tfrac{1}{2}-\beta)\Omega^2 \end{bmatrix} \quad (52)$$

where

$$D = [1+2\xi\Omega\gamma+(1+\alpha)\beta\Omega^2] \quad (53)$$

Note that when $\alpha = 0$ this reduces to the Newmark-β operator as expected.

Calculation of the invariants of A yields

$$A_1 = 1 - \frac{2\xi\Omega}{2D} - \frac{\Omega^2}{2D}\left[(1+\alpha)\left(\frac{1}{2}+\gamma\right) - \alpha\beta\right] \quad (54)$$

$$A_2 = 1 - \frac{2\xi\Omega}{D} - \frac{\Omega^2}{D}\left[2\alpha(\gamma-\beta)+\gamma-\frac{1}{2}\right] \quad (55)$$

$$A_3 = \frac{\Omega^2}{D}\left[\alpha\left(\beta-\gamma+\frac{1}{2}\right)\right] \quad (56)$$

while the local truncation error is found as

$$\sigma = (1/D)\{[2\xi\Omega(\tfrac{1}{2}-\gamma)]\omega^2 u + [\Omega(\gamma-\tfrac{1}{2}+\alpha) + 4\Omega\xi^2(\tfrac{1}{2}-\gamma)]\omega\dot{u}\} + O(\Delta t^2) \quad (57)$$

from which it follows that for second-order accuracy it is necessary that $\gamma = \tfrac{1}{2}$ and $\alpha = 0.5 - \gamma$ or $\alpha = 0$. The consequence of this constraint is that in general the α method cannot be second-order accurate unless it is reduced to the Newmark-β family of operators. An exception occurs when $\xi = 0$, in which case the α method is second-order accurate when $\alpha = 0.5 - \gamma$. For reasons to be made clear later, the stability requirements will be determined subject only to this one restraint (from the accuracy study) and the other requirement $\gamma = \tfrac{1}{2}$ will be temporarily neglected.

Employing Eqs. (30-34) the stability conditions are obtained as

$$-\tfrac{1}{2} \le \alpha \le 0, \qquad \beta \ge \tfrac{1}{4}, \qquad \tfrac{1}{2} \le \gamma \le 1 \quad (58)$$

subject to

$$\alpha + \gamma - \tfrac{1}{2} = 0 \quad \text{and} \quad \beta \ge \tfrac{1}{2}\gamma \quad (59)$$

These conditions insure second-order accuracy when $\xi = 0$.

By not requiring $\gamma = \tfrac{1}{2}$ when $\xi \neq 0$ the superior dissipative characteristics of this operator can be maintained with what is believed to be small accuracy loss in the low-frequency modes. This loss is expected to be small in most structural dynamics problems since $\xi \ll 1$ and since $|0.5-\gamma| \le \tfrac{1}{2}$ for stability. Thus, the first-order terms in σ (those that dictate $\gamma = \tfrac{1}{2}$) will be small because they are being multiplied by quantities that are much less than unity. The tradeoff between numerical dissipation and accuracy is considered to be worthwhile when dealing with multi-degree-of-freedom systems having many high-frequency modes that do not contribute significantly to the system response. Some numerical results illustrating the behavior of the spectral radius and the accuracy of the α-method operator are presented in Figs. 4-6.

The overshoot properties of the α methods are found in the same way as was described for the collocation methods and the results are the same for the general α method as for the Newmark method. Specifically,

$$d_1 \sim O(1)d_0 + O(\Omega)\xi d_0 + O(\Omega)\frac{\xi^2}{\omega}v_0 \quad (60)$$

$$v_0 \sim O(\Omega)\omega d_0 + O(\Omega)2\xi v_0 \quad (61)$$

which illustrates that system damping results in a displacement overshoot characteristic which is linear in Ω. This result has been confirmed by a numerical experiment showing significant overshoot in the displacement results for $\Omega > 1$. Hilber and Hughes' choice of $\beta = 0.25(0.5+\gamma)^2$ is validated by the numerical results for $\xi \ge 0$.

With regard to the algorithmic damping ratio, it is noted that the desirable characteristics exhibited in the undamped case are not present when system damping is introduced. The numerical results indicate that $\bar{\xi} \to \xi$ as $\Omega \to 0$ and that this is the maximum value attained since $\bar{\xi}$ decreases as Ω increases from zero. This means that in a multi-degree-of-freedom system with system damping the lowest frequency modes experience the greatest algorithmic damping, just as in the other operators.

The presence of system damping does not generally affect the relative period error except to decrease it slightly in relation to the undamped case. In both cases, increasing β while holding γ fixed results in an increase in the relative period error.

Newmark-β Results

If $\alpha = 0$, the α-method invariants become those for the Newmark-β operator and $A_3 \equiv 0$. Therefore, the characteristic

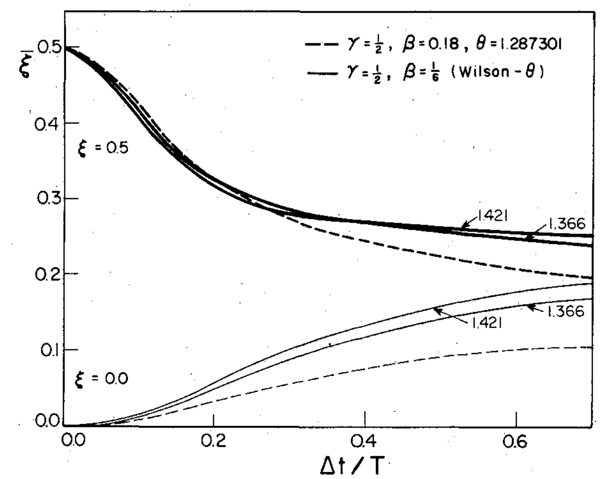

Fig. 3a Collocation methods algorithmic damping ratio behavior.

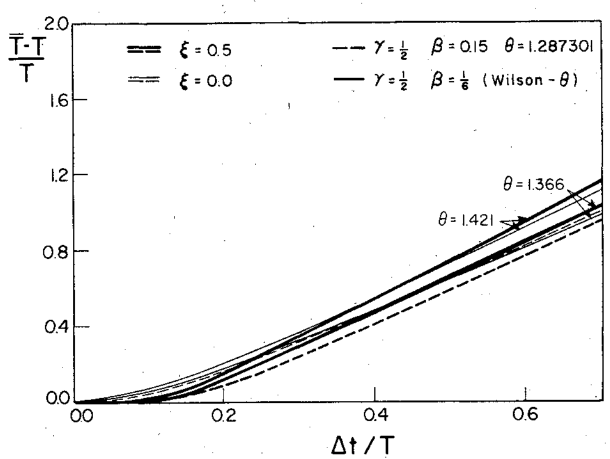

Fig. 3b Collocation methods relative period error behavior.

equation takes the special form

$$\lambda(\lambda^2 - 2A_1\lambda + A_2) = 0 \tag{62}$$

leaving only two roots that must lie within the unit circle. Following the same procedure as before, the necessary and sufficient conditions for spectral stability are found to be

$$1 + 2A_1 + A_2 > 0; \qquad 2 - 2A_1 > 0; \qquad 1 - 2A_1 + A_2 > 0 \tag{63}$$

These conditions imply that for the unconditional spectral stability of the Newmark-β method with $\xi \geq 0$,

$$\gamma \geq ½ \quad \text{and} \quad \beta \geq ½\gamma \tag{64}$$

This stability boundary has been identified for the undamped problem in Refs. 11 and 12 where it is also shown that $\beta = 0.25(0.5 + \gamma)^2$ is to be preferred for accuracy reasons. These stability domains are shown in Fig. 7. An accuracy investigation of the Newmark-β method shows that it is second-order accurate for $\gamma = ½$ regardless of the system damping. This is completely consistent with the results for the more general α method.

It also follows that the attributes of second-order accuracy and controllable numerical dissipation are mutually exclusive in the Newmark operator, since $\gamma = ½$ is required for second-order accuracy while $\gamma = ½$ allows no dissipation.[7] In contrast, unconditional stability and numerical dissipation can be achieved for $\gamma > ½$ but the method will only be first-order accurate. It is also noted that when $\beta = ¼$, $\gamma = ½$ the Newmark method corresponds to the trapezoidal rule, which has the least possible truncation error of all second-order methods.[1]

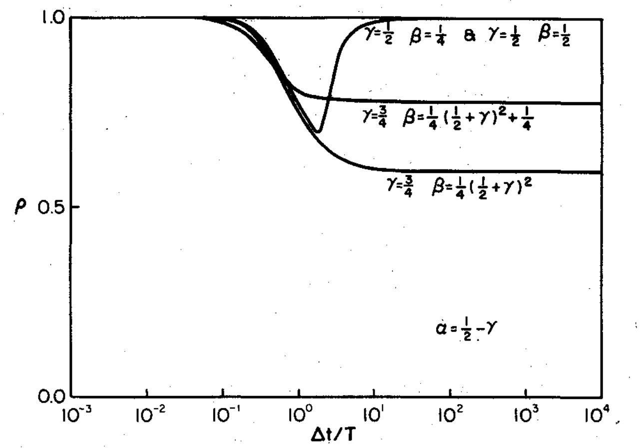

Fig. 4a α-method spectral radius behavior, $\xi = 0$.

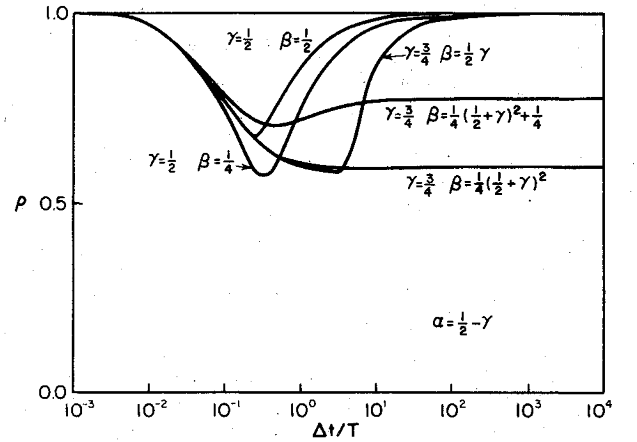

Fig. 4b α-method spectral radius behavior, $\xi = 0.5$.

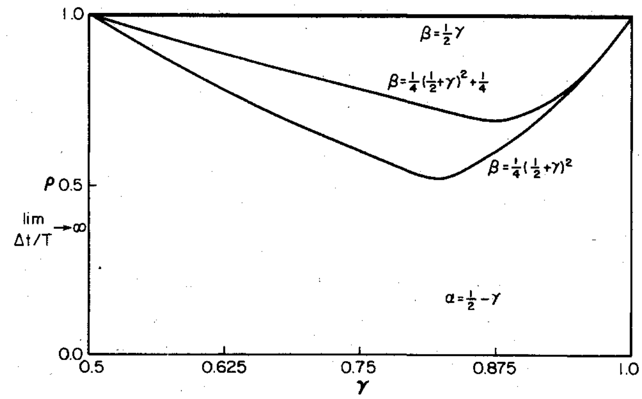

Fig. 4c α-method spectral radius behavior, $\Delta t/T \to \infty$.

In Ref. 2 it was shown that an operator's spectral radius is a minimum when the complex roots are about to bifurcate into a pair of real roots. The boundary in parameter space that satisfies this condition is defined by setting the discriminant of the characteristic equation equal to zero. For these optimal spectral properties a set of parameters should be chosen that result in the discriminant being slightly negative. This requirement insures that the discrete response be oscillatory in nature rather than critically damped or overdamped.

For the Newmark method ($A_3 = 0$) the discriminant is

$$\Delta = A_1^2 - A_2 \tag{65}$$

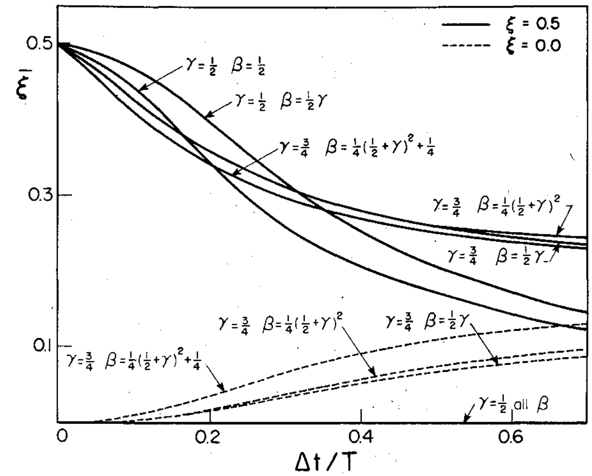

Fig. 5 **α-method algorithmic damping ratio behavior.**

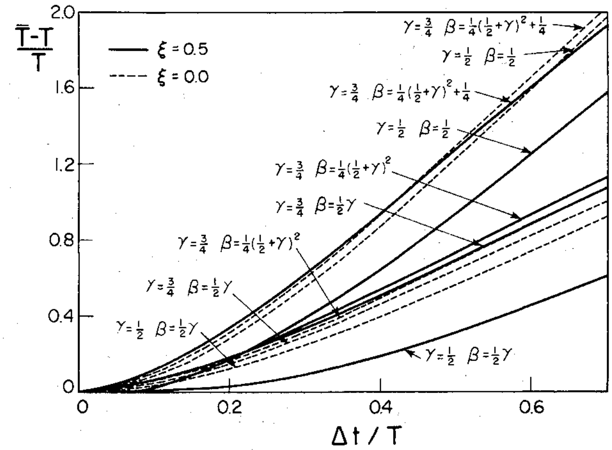

Fig. 6 **α-method relative period error behavior.**

which becomes

$$\Delta=\Omega^2[\tfrac{1}{4}(\tfrac{1}{2}+\gamma)^2-\beta]+\Omega\xi(\tfrac{1}{2}-\gamma)+(\xi^2-1) \tag{66}$$

The conditions that lead to $\Delta<0$ for all physically realizable Ω (i.e., $\Omega\geq 0$) are twofold. First, the quadratic in Ω must be concave downward, hence the condition

$$\beta>\tfrac{1}{4}(\tfrac{1}{2}+\gamma)^2 \tag{67}$$

Second, the roots of the quadratic in Ω must be complex or negative. From the solution for the roots $\Omega_{1,2}$,

$$\Omega_{1,2}=\{\tfrac{1}{2}\xi(\gamma-\tfrac{1}{2})\pm[\tfrac{1}{4}(\tfrac{1}{2}+\gamma)^2-\beta+\xi^2(\beta-\tfrac{1}{2}\gamma)]^{\frac{1}{2}}\}/[\tfrac{1}{4}(\tfrac{1}{2}+\gamma)^2-\beta] \tag{68}$$

it follows that the condition for complex roots is

$$\xi^2<[\beta-\tfrac{1}{4}(\tfrac{1}{2}+\gamma)^2]/(\beta-\tfrac{1}{2}\gamma) \tag{69}$$

while the roots will be negative if

$$\tfrac{1}{2}\xi(\gamma-\tfrac{1}{2})-[\tfrac{1}{4}(\tfrac{1}{2}+\gamma)^2-\beta+\xi^2(\beta-\tfrac{1}{2}\gamma)]^{\frac{1}{2}}>0 \tag{70}$$

Upon examining the stability boundaries, it is found that if $\beta=0.25(0.5+\gamma)^2$, then $\Delta<0$ only if $\gamma\geq\tfrac{1}{2}$ and $0\leq\xi<1$; also, if $0.5\gamma\leq\beta<0.25(0.5+\gamma)^2$ and $\gamma>0.5$, there will be one positive real root Ω_2 when $\Delta=0$ and $\Delta<0$ for all $\Omega<\Omega_2$.

In the case of the first-order accurate operators the behavior of ρ is similar to the $\xi=\tfrac{1}{2}$ case with $\beta=0.5\gamma$; however, when $\beta=0.25(0.5+\gamma)^2$, improved spectral properties result as $\Delta t/T\to\infty$. Although the choice of $\beta=0.25(0.5+\gamma)^2$ leads to the minimum value of ρ as $\Delta t/T\to\infty$, when $\xi\geq 0$ a dip develops in the ρ vs $\Delta t/T$ curve near $\Delta t/T=1$. This means that modes with frequencies in that range receive excessive numerical dissipation relative to higher-frequency modes. This is an undesirable trait that cannot be avoided but which must be borne in mind as the magnitude of this dip is directly proportional to the system damping.

The spectral radius ρ for each of the three curves discussed above is plotted as a function of γ in Fig. 8c for $\Delta t/T\to\infty$. The following observations should be made. When the lower stability limit is used to determine β (i.e., $\beta=0.5\gamma$), then ρ is unity for large $\Delta t/T$ for all values of γ. If β is chosen according to $\beta=0.25(0.5+\gamma)^2$, the least value of ρ ($\rho=0$) is obtained when $\gamma=1.5$ and corresponds to a triple root of the characteristic equation. Numerical experiments show that when $\xi\neq 0$ and $\gamma=1.5$, $\rho>0$, but still very much less than unity, and the spectral radius continues to move further away from zero as the damping is increased. When points above $\beta=0.25(0.5+\gamma)^2$ are chosen, curves such as those noted in Fig. 8c result and the final curve is for points midway between $\beta=0.5\gamma$ and $\beta=0.25(0.5+\gamma)^2$. The relevant aspect to note here is that points on the curve $\beta=0.25(0.5+\gamma)^2$ give the minimum values for ρ and all other points in the stability domain yield larger values of ρ. The curves in Fig. 8c will not vary with ξ for the reasons outlined previously.

In the special case of the Newmark method, the complex conjugate pair of roots are given by

$$\lambda_{2,3}=A_1\pm\sqrt{A_1^2-A_2},\qquad A_1^2-A_2<0 \tag{71}$$

Hence the algorithmic damping ratio for the Newmark method, based on the definitions from Ref. 3, is

$$\bar{\xi}=-\frac{\ln A_2}{2\bar{\Omega}}\quad\text{where}\quad\bar{\Omega}=\frac{\arctan(\sqrt{A_2-A_1^2}/A_1)}{\sqrt{1-\xi^2}} \tag{72}$$

Expressing A_1 and A_2 in terms of γ, β, ξ, and Ω from Eqs. (54-56), it can be shown that

$$\lim_{\Omega\to 0}\bar{\xi}=\xi \tag{73}$$

This result is illustrated in Fig. 9 where it is noted that the limit is independent of the values of γ and β. Further, the maximum value of $\bar{\xi}$ occurs when $\Omega\to 0$ only for the second-order cases where $\gamma=\tfrac{1}{2}$, while if $\gamma>\tfrac{1}{2}$ the maximum value of $\bar{\xi}$ is attained for some nonzero value of Ω.

The behavior of the spectral radius for the Newmark-β operator as a function of $\Delta t/T$ is shown in Fig. 8. These figures show that when $\xi=0$ and $\gamma=\tfrac{1}{2}$ there is no dissipation,

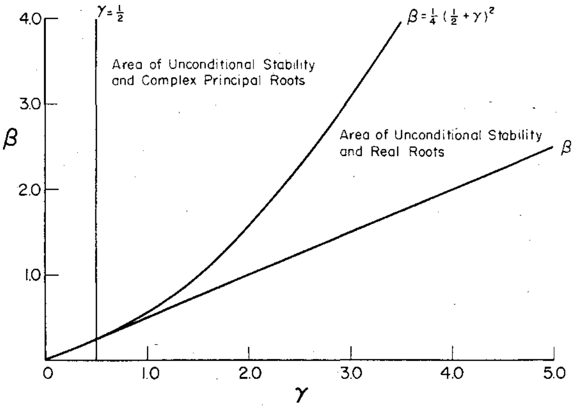

Fig. 7 **Newmark-β method stability boundaries.**

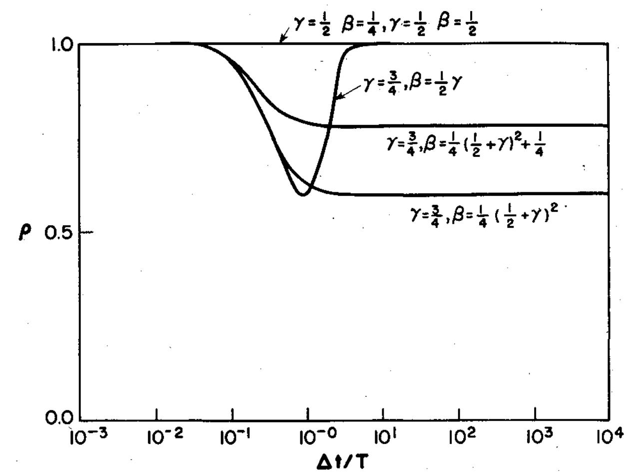

Fig. 8a Newmark-β method spectral radius behavior, $\xi=0$.

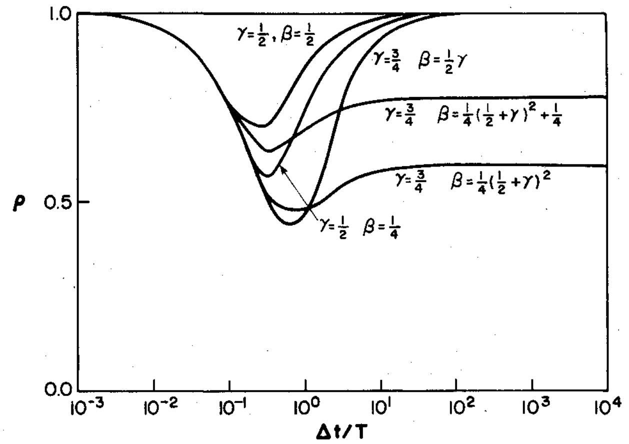

Fig. 8b Newmark-β method spectral radius behavior, $\xi=0.5$.

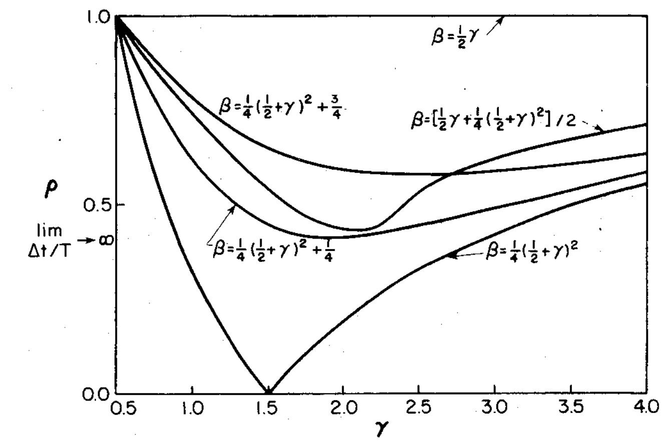

Fig. 8c Newmark-β method spectral radius behavior, $\Delta t/T\rightarrow\infty$.

but that when $\xi\neq 0$ there is some dissipation introduced in the neighborhood of the frequencies associated with $\Delta t/T=1$ and also that the minimum value of ρ for $\gamma=½$ decreases with increasing damping. In the limit as $\Delta t/T\rightarrow\infty$ it is seen that $\rho\rightarrow 1$ for all ξ, when $\gamma=½$. One other observation is that when $\gamma=½$ the minimum value of ρ is directly proportional to β (where $\beta=¼$ yields the minimum value), independent of the damping magnitude ($\xi>0$).

Numerical studies indicate that, for the Newmark method with constant γ, the algorithmic damping ratio decreases with increasing values of β and the relative period error increases

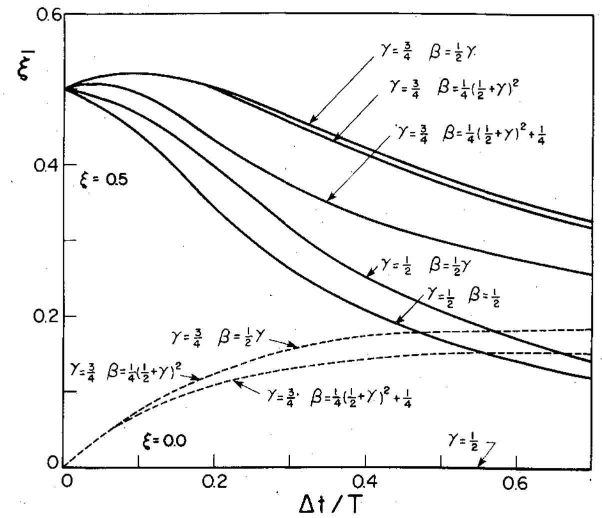

Fig. 9 Newmark-β method algorithmic damping ratio behavior.

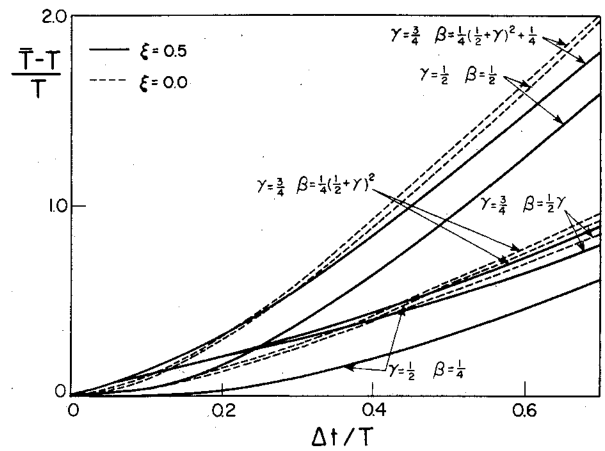

Fig. 10 Newmark-β method relative period error behavior.

with increasing β (Figs. 9 and 10), when $0\leq\xi<1$. Thus, when choosing values of γ and β a tradeoff between the two inaccuracies must be made.

Summary

The following points summarize the key findings in this work:

1) The stability of a temporal operator applied to a multi-degree-of-freedom system with nonproportional damping can be determined by investigating the spectral stability of the operator as applied to a single-degree-of-freedom system.

2) Satisfaction of the five stability conditions derived in this work is a necessary and sufficient condition for unconditional spectral stability of the operator.

3) The presence of system damping does not change the stability boundaries for any of the temporal operators investigated.

4) The general collocation, Wilson-θ, and Newmark-β methods can be second-order accurate for the damped and the undamped problems without changing the values of the integration parameters.

5) The alpha method can be second-order accurate for the undamped problem but is first-order accurate for the damped problem, unless it is reduced to the Newmark-β method.

6) The general collocation and Wilson-θ methods possess inherent quadratic overshoot properties that make them unattractive for application to multi-degree-of-freedom problems regardless of the amount of damping present in the system.

7) In the general collocation and Wilson-θ methods the choice of parameters to achieve the best spectral properties and the best error measure properties are mutually exclusive.

8) In the alpha and Newmark-β methods the choice of parameters to achieve the best spectral properties and the best error measure are mutually inclusive.

9) The best spectral properties for any of the operators investigated occur for choices of those integration parameters causing the discriminant to be negative and very near to zero.

10) In all the methods the introduction of system damping increases the algorithmic damping ratio, with the greatest increase being in the low-frequency modes.

11) The alpha method has superior spectral and dissipation properties compared to the other operators under any damping conditions.

Acknowledgment

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